Lecture 4: Mean-Variance Theory and CAPM

Bernardino Adao

ISEG, Lisbon School of Economics and Management

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Mean-Variance Theory (MVT)

- Economist Harry Markowitz introduced MVT in a 1952, for which he was awarded a Nobel Prize in 1990
- The assumption made by the theory is that only the mean and variance of the investment return matter
- The MVT is a method of constructing a portfolio that generates a maximum return for a given level of risk or a minimum risk for a stated return
- Although MVT has some limitations, it continues to be a cornerstone for portfolio managers
- **Remark:** Markets do not have to be complete

Reduction of Portfolio Risk Through Diversification

- Define a portfolio with return

$$R_p = \sum_{i=1}^{n} w_i R_i$$

- In general, higher correlation between assets in a portfolio will result in a higher portfolio variance.

$$\sigma_{R_p}^2 = \sum_{i=1}^{n} (w_i)^2 \sigma_{R_i}^2 + \sum_{i=1}^{n} \sum_{k \neq i} w_i w_k cov(R_i, R_k)$$

- If the weights are positive, a lower covariance between the securities implies lower portfolio variance.
- By combining various risky assets we can improve the portfolio's return-risk characteristics, resulting in a better trade-off.

Reduction of Portfolio Risk Through Diversification

- If the returns are uncorrelated, i.e. with covariance zero, the expression above reduces to:

$$\sigma^2_{R_p} = \sum_{i=1}^{n} (w_i)^2 \, \sigma^2_{R_i}$$

- To understand this more clearly, let us assume that we invest equal amounts in each asset

$$w_i = \frac{1}{n}$$

and $\sigma^2_{R_i}$ is the same for all assets, i.e. $\sigma^2_{R_i} = \sigma^2_R$

Reduction of Portfolio Risk Through Diversification

- Thus
$$\sigma^2_{R_p} = \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 \sigma^2_R = \frac{1}{n}\sigma^2_R$$
- As n gets larger, the variance tends to zero.
- In other words, the more independent assets in the portfolio, the lesser the variance and hence the risk.
- That is to say; risk can be minimized by diversification.

Calculating the Optimal Portfolio

- Consider a portfolio made up of assets A and B. Let σ_{R_A} and σ_{R_B} represent the standard deviation of the assets rate of return, respectively.
- Also, let w_A and let w_B represent the proportion of funds invested in each asset.
- We describe below how we can make the portfolio of the two assets optimal.
- An optimal portfolio minimizes risk for a given level of return.
- The optimal portfolio is created by investing the right proportion of funds in the respective assets making up the portfolio.
- To calculate the optimal portfolio, we, therefore, need to compute the appropriate asset allocations that ensure minimum risk.

Calculating the Optimal Portfolio

- The portfolio return is $R_P = w_A R_A + w_B R_B$ where the weights sum to one, $w_A + w_B = 1$
- To determine the asset allocation that minimizes portfolio risk, we differentiate the portfolio variance

$$\sigma^2_{R_P} = w_A^2 \sigma^2_{R_A} + (1 - w_A)^2 \sigma^2_{R_B} + 2 w_A (1 - w_A)\, cov(R_A, R_B)$$

  w.r.t. w_A
- Equate the first derivative to zero and solve for the minimum point
- The value w_A that minimizes portfolio risk, is

$$w_A = \frac{\sigma^2_{R_B} - cov(R_A, R_B)}{\sigma^2_{R_A} + \sigma^2_{R_B} - 2cov(R_A, R_B)}$$

Case 1: Without a risk free asset

- A portfolio is a **mean variance frontier portfolio** if it has the minimum variance among the portfolios that have the same expected rate of return.
- Let the vector of asset returns be $\mathbf{R}$.
- Let $\mathbf{E}$ be the vector of mean returns, $\mathbf{E} \equiv E(\mathbf{R})$, and
- Let Σ the variance-covariance matrix $\Sigma = E\left[(\mathbf{R} - \mathbf{E})(\mathbf{R} - \mathbf{E})^T\right]$.
- A portfolio is defined by its weights $\mathbf{w}$ on the securities.
- The portfolio return is $\mathbf{w}^T\mathbf{R}$ where the weights sum to one, $\mathbf{w}^T\mathbf{1} = \mathbf{1}$.

Case 1: Without a risk free asset

- The problem "**choose a portfolio to minimize variance for a given mean**" is then

$$\min_{\{\mathbf{w}\}} \left\{ \frac{1}{2} \mathbf{w}^T \Sigma \mathbf{w} \right\} \text{ s.t. } \mathbf{w}^T \mathbf{E} = \mu \text{ and } \mathbf{w}^T \mathbf{1} = 1.$$

- The objective function is convex in **w** and the restrictions are linear in **w**.
- Let the Lagrange multipliers on the constraints be λ and δ.

Case 1: Without a risk free asset

- Lagrangian

$$L = \frac{1}{2}\mathbf{w}^T\Sigma\mathbf{w} + \lambda\left(\mu - \mathbf{w}^T\mathbf{E}\right) + \delta\left(1 - \mathbf{w}^T\mathbf{1}\right)$$

- A first order condition is

$$\Sigma\mathbf{w} - \lambda\mathbf{E} - \delta\mathbf{1} = 0$$

$$\Rightarrow \mathbf{w} = \Sigma^{-1}(\lambda\mathbf{E} + \delta\mathbf{1}).$$

- Replacing in the constraints,

$$\mathbf{E}^T\mathbf{w} = \mathbf{E}^T\Sigma^{-1}(\lambda\mathbf{E} + \delta\mathbf{1}) = \mu$$

$$\mathbf{1}^T\mathbf{w} = \mathbf{1}^T\Sigma^{-1}(\lambda\mathbf{E} + \delta\mathbf{1}) = 1$$

or

$$\begin{bmatrix} \mathbf{E}^T\Sigma^{-1}\mathbf{E} & \mathbf{E}^T\Sigma^{-1}\mathbf{1} \\ \mathbf{1}^T\Sigma^{-1}\mathbf{E} & \mathbf{1}^T\Sigma^{-1}\mathbf{1} \end{bmatrix} \begin{bmatrix} \lambda \\ \delta \end{bmatrix} = \begin{bmatrix} \mu \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} \lambda \\ \delta \end{bmatrix} = \begin{bmatrix} \mu \\ 1 \end{bmatrix}$$

where $A = \mathbf{E}^T\Sigma^{-1}\mathbf{E}$; $B = \mathbf{E}^T\Sigma^{-1}\mathbf{1}$; $C = \mathbf{1}^T\Sigma^{-1}\mathbf{1}$

$$\begin{bmatrix} \lambda \\ \delta \end{bmatrix} = \begin{bmatrix} A & B \\ B & C \end{bmatrix}^{-1} \begin{bmatrix} \mu \\ 1 \end{bmatrix}$$

Case 1: Without a risk free asset

Remember: For any nonsingular matrix $\mathbf{A}$:
$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} adj(\mathbf{A})$, where $|\mathbf{A}|$ is the determinant of $\mathbf{A}$ and $adj(\mathbf{A})$ is defined as the transpose of the matrix $[A_{ij}]$, where A_{ij} is the cofactor of the element a_{ij}.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Hence,

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix}^{-1} = \frac{1}{AC - B^2} \begin{bmatrix} C & -B \\ -B & A \end{bmatrix}$$

$$\lambda = \frac{C\mu - B}{AC - B^2}$$

$$\delta = \frac{A - B\mu}{AC - B^2}$$

Case 1: Without a risk free asset

- Plugging in

$$\mathbf{w} = \Sigma^{-1}(\lambda \mathbf{E} + \delta \mathbf{1})$$

we get the portfolio weights

$$\mathbf{w} = \Sigma^{-1}\left(\frac{C\mu - B}{AC - B^2}\mathbf{E} + \frac{A - B\mu}{AC - B^2}\mathbf{1}\right)$$

$$\mathbf{w} = \Sigma^{-1}\frac{\mathbf{E}(C\mu - B) + \mathbf{1}(A - B\mu)}{AC - B^2}$$

to get the portfolio variance we compute

$$\Sigma \mathbf{w} = \frac{\mathbf{E}(C\mu - B) + \mathbf{1}(A - B\mu)}{AC - B^2}$$

Case 1: Without a risk free asset

- thus the portfolio variance is

$$\mathbf{w}^T\Sigma\mathbf{w} = \frac{\mathbf{w}^T\mathbf{E}(C\mu - B) + \mathbf{w}^T\mathbf{1}(A - B\mu)}{AC - B^2}$$

or

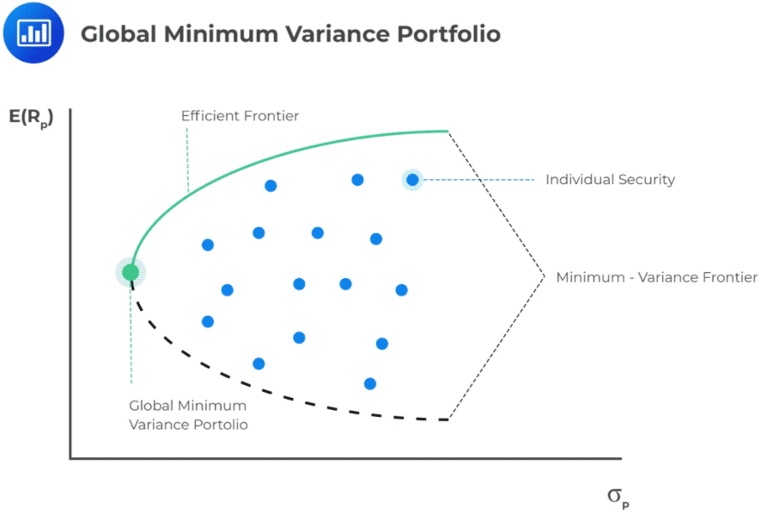
$$var(R_p) = \frac{C\mu^2 - 2B\mu + A}{AC - B^2}$$

since $\mathbf{w}^T\mathbf{E} = \mu$ and $\mathbf{w}^T\mathbf{1} = 1$.

- The variance is a quadratic function of the mean
- The variance is a parabola
- The standard deviation (volatility) is a hyperbola

Remark: The square root of a parabola is a hyperbola

Case 1: Without a risk free asset



Case 1: Without a risk free asset

- Example: Lets take

$$\Sigma = \begin{bmatrix} 1 & 0.25 \\ 0.25 & 1.2 \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} 1 \\ 1.2 \end{bmatrix}$$

$$A = \mathbf{E}^T \Sigma^{-1} \mathbf{E} = 1.7934$$

$$B = \mathbf{E}^T \Sigma^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1.6264$$

$$C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \Sigma^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1.4945$$

Let

$$y^2 = var(R_p) \text{ and } x = \mu$$

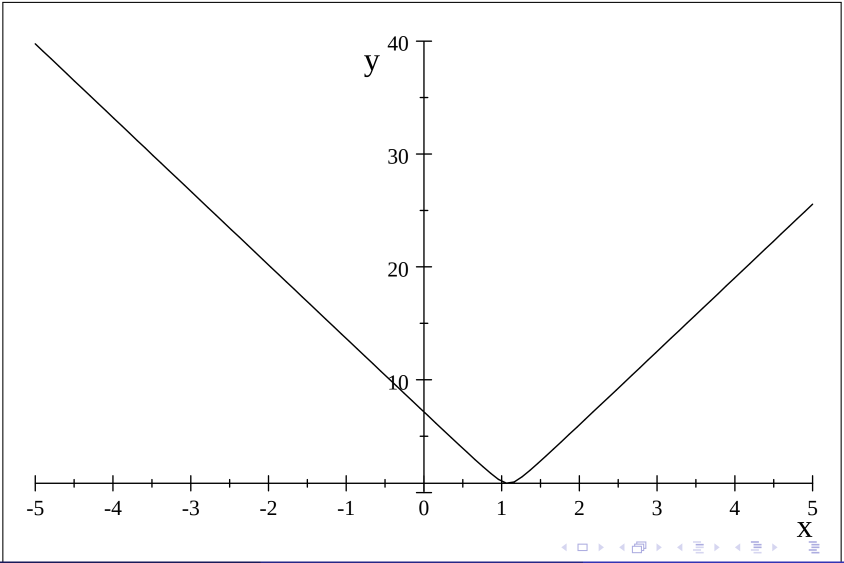
$$y^2 = \frac{Cx^2 - 2Bx + A}{AC - B^2}$$

$$y^2 = \frac{1.4945x^2 - 2(1.6264)x + 1.7934}{1.7934(1.4945) - (1.6264)^2}$$

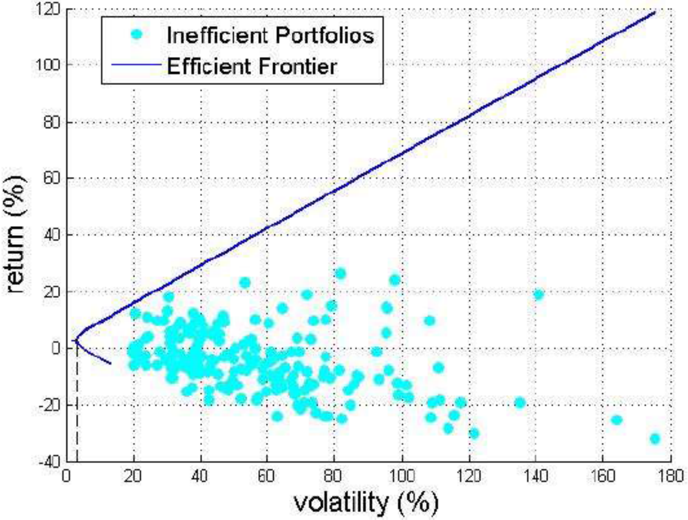
$$\begin{aligned} y &= \left(\frac{1.4945x^2 - 2(1.6264)x + 1.7934}{1.7934(1.4945) - (1.6264)^2}\right)^{1/2} \\ &= \sqrt{42.628x^2 - 92.780x + 51.153} \end{aligned}$$

Case 1: Without a risk free asset

Minimum variance portfolio (inverted)



Case 1: Without a risk free asset



Case 1: Without a risk free asset

- Example: If the mean is one: $x = 1$

The standard deviation is $y = \sqrt{42.628 - 92.780 + 51.153} = 1.0005$

- **minimum-variance portfolio**

solves the problem

$$\min_{\{\mu\}} \{var(R_p)\} = \frac{C\mu^2 - 2B\mu + A}{AC - B^2}$$

FOC

$$\frac{2C\mu - 2B}{AC - B^2} = 0$$

$$\mu^{\text{min var}} = B/C$$

Case 1: Without a risk free asset

As

$$\mathbf{w} = \Sigma^{-1} \frac{\mathbf{E}(C\mu - B) + \mathbf{1}(A - B\mu)}{AC - B^2}$$

The weights of the minimum variance portfolio are

$$\mathbf{w} = \Sigma^{-1} \frac{\mathbf{E}(C\frac{B}{C} - B) + \mathbf{1}(A - B\frac{B}{C})}{AC - B^2}$$

$$\mathbf{w} = \Sigma^{-1} \frac{\mathbf{1}\frac{1}{C}(AC - B^2)}{AC - B^2}$$

$$\mathbf{w} = \Sigma^{-1}\mathbf{1}/(\mathbf{1}^T\Sigma^{-1}\mathbf{1})$$

Case 1: Without a risk free asset

- When does a mean-variance frontier exist? As long as Σ^{-1} exists.
- Theorem: So long as the variance-covariance matrix of returns is non-singular, there is a mean-variance frontier.
- This rules out two returns perfectly correlated but yielding different means

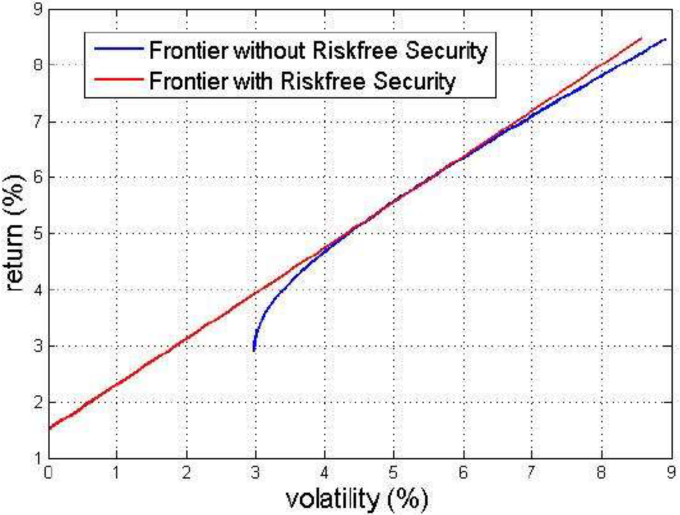
Case 1: Without a risk free asset

- *Result:* **The frontier is spanned by any two frontier returns.**
- Since **w** is a linear function of μ,

$$\mathbf{w} = \Sigma^{-1} \frac{\mathbf{E}(C\mu - B) + \mathbf{1}(A - B\mu)}{AC - B^2}$$

- A portfolio on the frontier with mean $\mu_3 = \lambda\mu_1 + (1-\lambda)\mu_2$ can be achieved with a portfolio of weights λ and $(1-\lambda)$ on two distinct portfolios on the frontier with mean returns μ_1 and μ_2.
- The weights on the third portfolio are given by $\mathbf{w}_3 = \lambda\mathbf{w}_1 + (1-\lambda)\mathbf{w}_2$, because **w** is linear in μ.

Case 2: With a risk free asset



Case 2: With a risk free asset

- assume now that there is a risk free asset with return R_f
- let $\mathbf{w} = (w_1, w_2, ..., w_n)$ be the vector of weights on the n risky assets and so that $1 - \sum_{i=1}^{n} w_i$ is the weight on the risk free security
- the investor's optimization problem is

$$\min_{\{\mathbf{w}\}} \left\{ \frac{1}{2} \mathbf{w}' \Sigma \mathbf{w} \right\} \text{ s.t. } \left(1 - \sum_{i=1}^{n} w_i \right) R_f + \mathbf{w}' \mathbf{E} = E(R_P)$$

the lagrangian

$$L = \frac{1}{2} \mathbf{w}' \Sigma \mathbf{w} + \lambda \left(E(R_P) - R_f - \mathbf{w}' (\mathbf{E} - R_f \mathbf{1}) \right)$$

a FOC is

$$\Sigma \mathbf{w} = \lambda (\mathbf{E} - R_f \mathbf{1})$$

solving for $\mathbf{w}$

$$\mathbf{w} = \lambda \Sigma^{-1} (\mathbf{E} - R_f \mathbf{1})$$

Case 2: With a risk free asset

- replacing in $\sigma^2(R_p) \equiv \mathbf{w}'\Sigma\mathbf{w}$

$$\sigma^2(R_p) = \lambda^2 (\mathbf{E} - R_f\mathbf{1})' \Sigma^{-1} (\mathbf{E} - R_f\mathbf{1})$$

- solving for λ

$$\lambda = \frac{\sigma(R_p)}{\sqrt[2]{(\mathbf{E} - R_f\mathbf{1})' \Sigma^{-1} (\mathbf{E} - R_f\mathbf{1})}}$$

- replacing the λ in the FOC we get the optimal weights

$$\mathbf{w} = \frac{\sigma(R_p)}{\sqrt[2]{(\mathbf{E} - R_f\mathbf{1})' \Sigma^{-1} (\mathbf{E} - R_f\mathbf{1})}} \Sigma^{-1} (\mathbf{E} - R_f\mathbf{1})$$

  are proportional to $\Sigma^{-1} (\mathbf{E} - R_f\mathbf{1})$

- **Remark:** $\frac{\sigma(R_p)}{\sqrt[2]{(\mathbf{E} - R_f\mathbf{1})' \Sigma^{-1} (\mathbf{E} - R_f\mathbf{1})}}$ is a scalar

Case 2: With a risk free asset

- the tangency portfolio, $\mathbf{w}_t$, is such that:

$$\mathbf{1}'\mathbf{w}_{,} = 1$$

- thus must find κ such that

$$\mathbf{1}'\kappa\Sigma^{-1}\left(\mathbf{E} - R_f\mathbf{1}\right) = 1 \Longrightarrow \kappa = 1/\mathbf{1}'\Sigma^{-1}\left(\mathbf{E} - R_f\mathbf{1}\right)$$

$$\mathbf{w}_m = \frac{\Sigma^{-1}\left(\mathbf{E} - R_f\mathbf{1}\right)}{\mathbf{1}'\Sigma^{-1}\left(\mathbf{E} - R_f\mathbf{1}\right)}$$

- the expected value of the **tangency portfolio**, $\mathbf{w}_m'\mathbf{E}$

$$E\left(R_m\right) = \frac{\left(\mathbf{E} - R_f\mathbf{1}\right)'\Sigma^{-1}}{\mathbf{1}'\Sigma^{-1}\left(\mathbf{E} - R_f\mathbf{1}\right)}\mathbf{E}$$

- and the standard deviation of the **tangency portfolio** $\sqrt[2]{\mathbf{w}_m'\Sigma\mathbf{w}_m}$

$$\sigma(R_m) = \frac{\sqrt[2]{\left(\mathbf{E} - R_f\mathbf{1}\right)'\Sigma^{-1}\left(\mathbf{E} - R_f\mathbf{1}\right)}}{\mathbf{1}'\Sigma^{-1}\left(\mathbf{E} - R_f\mathbf{1}\right)}$$

Case 2: With a risk free asset

- any efficient portfolio is a combination of the tangency portfolio and the risk-free asset:

$$\begin{aligned} \mathbf{w} &= \theta \mathbf{w}_m, \text{ with } \theta \geq 0 \\ & \quad \text{and } 1 - \theta \mathbf{1}' \mathbf{w}_m \text{ on the riskless asset} \end{aligned}$$

- by definition

$$E(R_P) = R_f + \mathbf{w}_p' (\mathbf{E} - R_f \mathbf{1})$$

when we replace the $\mathbf{w}'$ we get the **Capital Market Line** (CML)

- recall that

$$\begin{aligned} \mathbf{w}_p' &= \frac{\sigma(R_p) (\mathbf{E} - R_f \mathbf{1})' \Sigma^{-1}}{\sqrt[2]{(\mathbf{E} - R_f \mathbf{1})' \Sigma^{-1} (\mathbf{E} - R_f \mathbf{1})}} \\ &= \frac{\sigma(R_p)}{\sigma(R_m)} \frac{(\mathbf{E} - R_f \mathbf{1})' \Sigma^{-1}}{\mathbf{1}' \Sigma^{-1} (\mathbf{E} - R_f \mathbf{1})} \end{aligned}$$

Case 2: With a risk free asset

- get

$$
\begin{aligned}
E\left(R_P\right) &= R_f + \frac{\sigma(R_p)\left(\mathbf{E}-R_f\mathbf{1}\right)'\Sigma^{-1}}{\sigma(R_m)\mathbf{1}'\Sigma^{-1}\left(\mathbf{E}-R_f\mathbf{1}\right)}\left(\mathbf{E}-R_f\mathbf{1}\right) \\
&= R_f + \frac{\sigma(R_p)\left(\mathbf{E}-R_f\mathbf{1}\right)'\Sigma^{-1}}{\sigma(R_m)\mathbf{1}'\Sigma^{-1}\left(\mathbf{E}-R_f\mathbf{1}\right)}\mathbf{E} \\
&\quad -\frac{\sigma(R_p)\left(\mathbf{E}-R_f\mathbf{1}\right)'\Sigma^{-1}}{\sigma(R_m)\mathbf{1}'\Sigma^{-1}\left(\mathbf{E}-R_f\mathbf{1}\right)}R_f\mathbf{1} \\
&= R_f + \frac{\sigma(R_p)}{\sigma(R_m)}\left(E\left(R_m\right)-R_f\right)
\end{aligned}
$$

Case 2: With a risk free asset

- The CML represents the risk-return tradeoff:

$$E\left(R_p\right) = R_f + \frac{\sigma(R_p)}{\sigma(R_m)}\left(E\left(R_m\right) - R_f\right)$$

- **Conclusion:**
- The **tangency portfolio** is the key risky portfolio, and the mean-variance efficient frontier is determined by leveraging this portfolio with the risk-free asset.
- **Exercise:** Show that the **tangency portfolio** (also called the **market portfolio** or optimal risky portfolio) is the portfolio of risky assets that maximizes the **Sharpe ratio**, $\frac{E(R_p)-R_f}{\sigma(R_p)}$.

Case 2: With a risk free asset

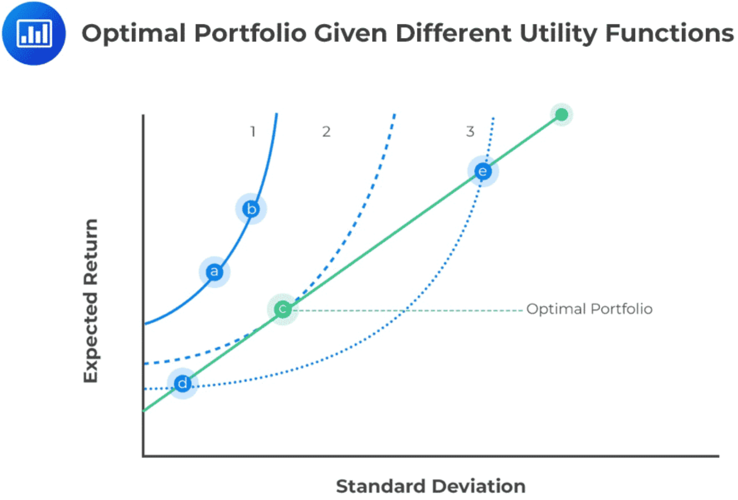
- **Exercise:** The risk-free rate is either below or at the point of minimum variance on the risky frontier. Why can't it be above? Explain
- Let $R_f < \mu^{\text{min var}}$
- The efficient frontier becomes a straight line that is tangential to the risky efficient frontier and with the y-intercept equal to R_f
- This result says that every investor optimally choose to invest in a combination of risk-free security and a single risky portfolio, i.e. the tangency portfolio

- **Exercise:** Show that the efficient frontier is indeed a straight line
- Hint: consider forming a portfolio of the risk-free with any risky portfolio. Show that the mean and standard deviation of the portfolio varies linearly with a share α, where α is the weight on the risky security
- **Exercise:** Describe the efficient frontier if no borrowing is allowed

Optimal Portfolio

- Assume the investor is a mean-variance optimizer: $U(\mu, \sigma)$
- The investor's indifference curves have a positive slope on the space (μ, σ) because investors are willing to hold a portfolio with more risk only if the portfolio expected return is higher
- The investor's indifference curves are convex because in order to accept equal increases in risk, the investor requires larger increases in the expected return, the higher the risk of his portfolio
- By imposing the investor's indifference curves (with negative slope and convex) on the space (μ, σ) we get the optimal portfolio

Optimal Portfolio



CAPM

Summing-up:

- Each investor will hold the same tangency portfolio of risky securities in conjunction with a position in the risk-free asset
- Because the **tangency portfolio** is held by all investors it is also called the **market portfolio**
- The **efficient frontier** is termed the **capital market line**

CAPM

- Let R_m and $E[R_m]$ denote the return and expected return, respectively, of the market, i.e. tangency portfolio
- The central insight of the CAPM is that in equilibrium the **riskiness of an asset** is not measured by the standard deviation of its return but by its **beta**
- In particular, there is a linear relationship between the expected return of asset i, $E[R_i]$, and the expected return of the market portfolio

$$E[R_i] = R_f + \beta_i \left[E[R_m] - R_f\right]$$

  where $\beta_i = Cov(R_i, R_m) / Var(R_m)$
- The **Security Market Line** (SML) (derived from **CAPM**) is also a straight line but plots expected return against beta, applying to all assets (efficient or inefficient).

CAPM

- $E\left[R_m\right] - R_f$ is the market risk premium (excess return of the market over the risk-free rate)
- Beta of asset i, β_i, measures its sensitivity to market movements
- If $\beta_i > 1$, the asset is more volatile than the market
- If $\beta_i < 1$,the asset is less volatile than the market
- If $\beta_i = 0$, the asset is uncorrelated with the market (e.g., risk-free asset)
- The CAPM states that investors should be compensated for both the time value of the funds (via R_f) and the risk taken (via β_i)

CAPM and CML

- The CML applies only to portfolios that are fully diversified (efficient portfolios)
- The CAPM applies to all assets, providing a pricing model for individual securities
- The CML uses total risk (standard deviation), while CAPM uses only systematic risk (beta)
- The slope of the CML is the Sharpe Ratio, $\frac{E(R_m)-R_f}{\sigma(R_m)}$, while the slope of the SML (CAPM) is the market risk premium, $E\left[R_m\right]-R_f$

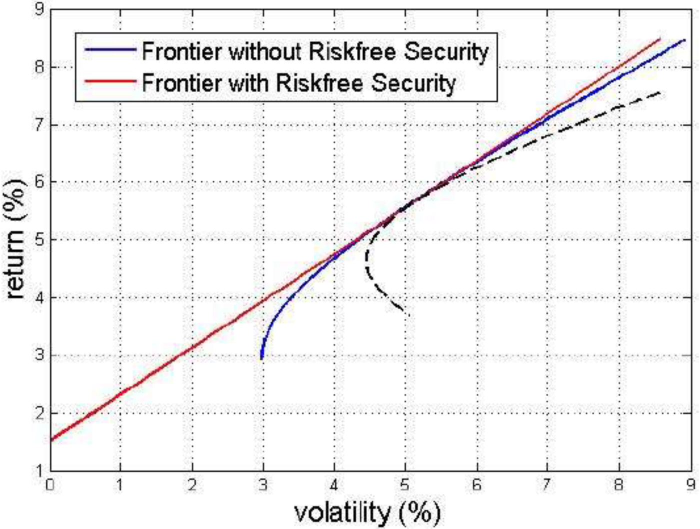
- Consider a portfolio with weights α and weight $1-\alpha$ on the risky security and market portfolio, respectively.
- Let R_α denote the (random) return of this portfolio as a function of α

$$E[R_\alpha] = \alpha E[R_i] + (1-\alpha)E[R_m]$$

$$\sigma^2_{R_\alpha} = \alpha^2\sigma^2_{R_i} + (1-\alpha)^2\sigma^2_{R_m} + 2\alpha(1-\alpha)cov(R_i, R_m)$$

- As α varies, the mean and standard deviation, $(E[R_\alpha], \sigma^2_{R_\alpha})$ trace out a curve that cannot (why?) cross the efficient frontier. This curve is depicted as the dashed curve below.

CAPM



Frontier without Riskfree Security
Frontier with Riskfree Security
return (%)
volatility (%)
1
2
3
4
5
6
7
8
9
0
1
2
3
4
5
6
7
8
9

CAPM

- At $\alpha = 0$ this curve must be tangent to the capital market line.

Therefore the slope of the curve at $\alpha = 0$ must equal the slope of the capital market line. Using the equations for $E[R_\alpha]$ and $\sigma^2_{R_\alpha}$ we see the slope is given by

$$\frac{dE[R_\alpha]}{d\sigma_{R_\alpha}} = \frac{\frac{dE[R_\alpha]}{d\alpha}}{\frac{d\sigma_{R_\alpha}}{d\alpha}}$$

where

$$\frac{dE[R_\alpha]}{d\alpha} = E\left[R_i\right] - E\left[R_m\right]$$

$$\frac{d\sigma_{R_\alpha}}{d\alpha} = \frac{1}{2}\left(\sigma^2_{R_\alpha}\right)^{-\frac{1}{2}}\left[2\alpha\sigma^2_{R_i} - 2(1-\alpha)\sigma^2_{R_m} + (2-4\alpha)cov(R_i, R_m)\right]$$

CAPM

(evaluate the derivatives when $\alpha = 0$)

$$\frac{dE[R_\alpha]}{d\alpha} = E[R_i] - E[R_m]$$

$$\frac{d\sigma_{R_\alpha}}{d\alpha} = (\sigma_{R_m})^{-1}\left[-\sigma_{R_m}^2 + cov(R_i, R_m)\right]$$

$$\frac{dE[R_\alpha]}{d\sigma_{R_\alpha}} = \frac{\frac{dE[R_\alpha]}{d\alpha}}{\frac{d\sigma_{R_\alpha}}{d\alpha}} = \frac{\sigma_{R_m}(E[R_i] - E[R_m])}{-\sigma_{R_m}^2 + cov(R_i, R_m)}$$

The slope of the capital market line is $(E[R_m] - R_f)/\sigma_{R_m}$

Equating the two

$$\frac{\sigma_{R_m}(E[R_i] - E[R_m])}{-\sigma_{R_m}^2 + cov(R_i, R_m)} = \frac{E[R_m] - R_f}{\sigma_{R_m}}$$

rearranging gives

$$E[R_i] = R_f + \beta_i\left[E[R_m] - R_f\right]$$

CAPM and SDF

Using the definition of SDF:

$$E[mR_i] = R_f E[m]$$

using the covariance formula

$$cov(m, R_i) + E[m]\,E[R_i] = R_f E[m]$$

and rearranging gives

$$E[R_i] = R_f - \frac{cov(m, R_i)}{E[m]}$$

CAPM

- The CAPM result is one of the most famous results in finance and, even though it arises from a simple one-period model, it provides an important insight to the problem of asset-pricing.
- For example, it is well-known that riskier securities should have higher expected returns in order to compensate investors for holding them.
- But how do we measure risk? Counter to the prevailing wisdom at the time the CAPM was developed, the riskiness of a security is not measured by its return volatility. Instead it is measured by its beta, which is proportional to its covariance with the market portfolio.
- **Exercise**: Why doesn't the CAPM result contradict the mean-variance problem formulation where investors do measure a portfolio's risk by its variance?

CAPM

- Many empirical research papers have been written to test the CAPM.
- For instance according to the Fama-MacBeth approach:
- Estimate the beta for each asset by doing the time series regressions:

$$R_i - R_f = \alpha + \beta_i (R_m - R_f) + \epsilon_i$$

where ϵ_i is the idiosyncratic or residual risk which is assumed to be independent of R_m

- Do the cross-section regression of the excess return $R_i - R_f$ of all assets on their betas:

$$R_i - R_f = \gamma_0 + \gamma_1 \beta_i + \epsilon_i$$

CAPM

$$R_i - R_f = \gamma_0 + \gamma_1 \beta_i + \epsilon_i$$

- For the CAPM to hold
  - then the intercept, γ_0, should be zero,
  - the slope, γ_1, should be statistically significant
  - $\gamma_1 > 1$ so that higher beta assets earn higher returns
- To check statistical significance:
  - The average of γ_1 across time is tested for significance.
  - If γ_1 is statistically significant (p-value < 0.05), then beta significantly explains returns, supporting CAPM.
  - If γ_1 is not significant (p-value > 0.05), it suggests that beta does not explain returns well, contradicting CAPM.

CAPM

- Summing-up:
- If beta significantly explains returns, it means higher beta stocks earn higher returns as CAPM predicts.
- If beta is insignificant, it suggests that other factors (e.g., size, value) might drive returns more than beta.
- Empirical studies (e.g., Fama & French 1992) found that beta alone does not fully explain stock returns, leading to multi-factor models.

CAPM

- The CAPM is an example of a so-called 1-factor model with the market return playing the role of the single factor.
- Fama and French (1992) found that size (SMB) and value (HML) factors explain returns better than beta alone.
- The Fama-French Three-Factor Model is:

$$E\left[R_i\right] = R_f + \beta_i \left[E\left[R_m\right] - R_f\right] + s_i SMB + h_i HML$$

where one of the factors is the market return, *SMB* refers to size and *HML* refers to value

- Empirical studies show that CAPM often fails because:
  - Small-cap stocks tend to outperform what CAPM predicts.
  - High book-to-market (value) stocks earn higher returns than expected.
  - Beta alone is not enough to explain asset returns.